NONSINGULARITY OF THE SAMPLE COVARIANCE MATRIX

by

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Technical Report No. 162

April 1971

University of Minnesota
Minneapolis, Minnesota

*This research was supported by U.S. Army Grant DA-ARO-D-31-124-71-G99.

Introduction.

Let $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n$ be $p \times 1$ random vectors. Define $A = \sum_{\alpha=1}^{N} (\underset{\sim}{X}_\alpha - \bar{\underset{\sim}{X}})(\underset{\sim}{X}_\alpha - \bar{\underset{\sim}{X}})'$, where $\bar{\underset{\sim}{X}} = \sum_{\alpha=1}^{N} \underset{\sim}{X}_\alpha / N$. Dykstra [1] proved that A is nonsingular with probability 1 when $N > p$ and $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_N$ is a sample from a nonsingular p-variate normal distribution. In this note, the question of nonsingularity of A (with probability 1) is studied under a more general set-up and later a general (matrix) quadratic form in $\underset{\sim}{X}_\alpha$'s is studied from this viewpoint.

Results.

Note that $A = \underline{\bar{X}}\, C_N\, \underline{\bar{X}}'$, where

$$\underline{\bar{X}} = (\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_N), \quad C_N = I_N - \frac{1}{N} J_N,$$

I_N is the $N \times N$ identity matrix and J_N is the $N \times N$ matrix with all elements equal to 1. Since rank $C_N = N - 1$ and $\mathrm{rank}(A) \leq \mathrm{rank}(C_N)$, A is singular if $p > N - 1$. In the sequel, we thus assume $p \leq N - 1$.

Theorem 1.

Suppose there exists a $(p-1)$-dimensional hyperplane L in R^p given by $\underset{\sim}{b}'\underset{\sim}{x} = a$ such that $\mathrm{Prob}[\underset{\sim}{X}_\alpha \in L, \alpha = 1, \ldots, N] > 0$ where $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_N$ are $p \times 1$ random vectors. Then $\mathrm{Prob}[A$ is singular$] > 0$.

Proof:

$$\underset{\sim}{X}_\alpha \in L,\ \alpha = 1, \ldots, N \Rightarrow \underset{\sim}{b}'A\underset{\sim}{b} = a^2 \underset{\sim}{e}_N' C_N \underset{\sim}{e}_N = 0$$

$$\Rightarrow A \text{ is singular}$$

$\underset{\sim}{e}_N$ is the $N \times 1$ vector with all elements equal to 1.

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Results.

Note that $A = \underline{X}\, C_N\, \underline{X}'$, where

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Suppose there exists a $(p-1)$-dimensional hyperplane L in R^p given by $b'x = a$ such that $\operatorname{Prob}[X_\alpha \in L,\ \alpha = 1, \dots, N] > 0$ where $X_1, \dots, X_N$ are $p \times 1$ random vectors. Then $\operatorname{Prob}[A$ is singular$] > 0$.

Proof:

$$X_\alpha \in L,\ \alpha = 1, \dots, N \Rightarrow b'Ab = a^2 e_N' C_N e_N = 0$$

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e_N is the $N \times 1$ vector with all elements equal to 1.

*This research was supported by U.S. Army Grant DA-ARO-D-31-124-71-G59.

Corollary 1.1.

Prob(A is singular) = 0 implies that for every (p-1)-dimensional hyperplane L, Prob($\underset{\sim}{X}_\alpha \in L$, $\alpha = 1,\ldots, N$) = 0 which is equivalent to Prob($\underset{\sim}{X}_\alpha \in L$) = 0 when $\underset{\sim}{X}_\alpha$'s are i.i.d.

Lemma.

Let $\underset{\sim}{X}_1,\ldots, \underset{\sim}{X}_p$ be $p \times 1$ mutually independent random vectors. Assume that for any $\alpha (\alpha = 1,\ldots, p)$ and any (p-1)-dimensional subspace L of R^p, Prob($\underset{\sim}{X}_\alpha \in L$) = 0. Then Prob($\underline{\overline{X}}_{[p]}$ is singular) = 0 where

$$\underline{\overline{X}}_{[p]} = (\underset{\sim}{X}_1,\ldots, \underset{\sim}{X}_p).$$

Proof:

We shall prove this by induction on p. Let $X'_\alpha = (X_{\alpha 1},\ldots, X_{\alpha p})$ and $\underset{\sim}{X}'_{\alpha[i]} = (X_{\alpha 1},\ldots, X_{\alpha i})$, $\underline{\overline{X}}_{[i]} = (X_{1[i]},\ldots, X_{i[i]})$, $i = 1,\ldots, p$. The condition in the lemma implies that Prob($\underset{\sim}{X}_{\alpha[i]} \in L_i$) = 0 for any (i-1)-dimensional subspace L_i of R^i $i = 1,\ldots, p$. Clearly Prob($\underline{\overline{X}}_{[1]}$ is singular) = 0. Assume, for $1 \leq r < p$, Prob($\underline{\overline{X}}_{[r]}$ is singular) = 0. Suppose, Prob($\underline{\overline{X}}_{[r+1]}$ is singular) > 0. Then there must exist a set of positive probability measure in the space of $\underset{\sim}{X}_{\alpha[r+1]}$, $\alpha = 1,\ldots, r$ for which

$$\text{Prob}[\det(\underset{\sim}{x}_{1[r+1]} \cdots \underset{\sim}{x}_{r[r+1]}, X_{r+1[r+1]})$$

$$= 0 | X_{\alpha[r+1]} = x_{\alpha[r+1]}, \alpha = 1,\ldots, r] > 0.$$

However, the above determinant may be written as $\underset{\sim}{b}'X_{r+1,[r+1]} = 0$, where $\underset{\sim}{b}$ is a function of $\underset{\sim}{x}_{\alpha[r+1]}$, $\alpha = 1,\ldots, r$ and the above conditional probability is the same as the unconditional probability which is zero by assumption unless $\underset{\sim}{b} = \underset{\sim}{0}$, and, in particular, the last component of $\underset{\sim}{b}$ is 0 which is the same as saying $\det(\underset{\sim}{x}_{1[r]},\ldots, \underset{\sim}{x}_{r[r]}) = 0$ But the probability of such event, by the induction hypothesis is 0. Hence Prob($\underline{\overline{X}}_{[r]}$ is singular) = 0.

Corollary 1.1.

Prob(A is singular) = 0 implies that for every $(p-1)$-dimensional hyperplane L, $\text{Prob}(\underset{\sim}{X}_r \in L,\ r = 1, \ldots, N) = 0$ which is equivalent to $\text{Prob}(\underset{\sim}{X}_r \in L) = 0$ when $\underset{\sim}{X}_r$'s are i.i.d.

Lemma.

Let $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_p$ be $p \times 1$ mutually independent random vectors. Assume that for any $r (r = 1, \ldots, p)$ and any $(p-1)$-dimensional subspace L of R^p, $\text{Prob}(\underset{\sim}{X}_r \in L) = 0$. Then $\text{Prob}(\overline{X}_{[p]}$ is singular$) = 0$ where

$$\overline{X}_{[p]} = (\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_p).$$

Proof:

We shall prove this by induction on p. Let $X_r' = (X_{r1}, \ldots, X_{rp})$ and $\underset{\sim}{X}'_{[i]} = (X_{q1}, \ldots, X_{qi})$, $\overline{X}_{[i]} = (\underset{\sim}{X}_{1[i]}, \ldots, \underset{\sim}{X}_{i[i]})$, $i = 1, \ldots, p$. The condition in the lemma implies that $\text{Prob}(\underset{\sim}{X}_{r[i]} \in L_i) = 0$ for any $(i-1)$-dimensional subspace L_i of R^i $i = 1, \ldots, p$. Clearly $\text{Prob}(\overline{X}_{[1]}$ is singular$) = 0$. Assume, for $1 \leq r < p$, $\text{Prob}(\overline{X}_{[r]}$ is singular$) = 0$. Suppose, $\text{Prob}(\overline{X}_{[r+1]}$ is singular$) > 0$. Then there must exist a set of positive probability measure in the space of $\underset{\sim}{X}_{q[r+1]}$, $q = 1, \ldots, r$ for which

$$\text{Prob}[\det(\underset{\sim}{X}_{1[r+1]} \ldots \underset{\sim}{X}_{r[r+1]}, X_{r+1[r+1]}) = 0 \mid X_{q[r+1]} = x_{q[r+1]}, q=1, \ldots,$$

$$= 0 | X_{q[r+1]} = x_{q[r+1]},\ q = 1, \ldots, r] > 0.$$

However, the above determinant may be written as $\underset{\sim}{b}' \underset{\sim}{X}_{r+1,[r+1]} = 0$, where $\underset{\sim}{b}$ is a function of $\underset{\sim}{X}_{q[r+1]}$, $q = 1, \ldots, r$ and the above conditional probability is the same as the unconditional probability which is zero by assumption unless $\underset{\sim}{b} = \underset{\sim}{0}$, and in particular, the last component of $\underset{\sim}{b}$ is 0 which is the same as saying $\det(\underset{\sim}{X}_{1[r]}, \ldots, \underset{\sim}{X}_{r[r]}) = 0$. But the probability of such event, by the induction hypothesis is 0. Hence $\text{Prob}(\overline{X}_{[r]}$ is singular$) = 0$.

Theorem 2.

Suppose $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_N$ are mutually independent and for every $\alpha (\alpha = 1, \ldots, N)$ and for every $(p-1)$-dimensional hyperplane L in R^p, $\text{Prob}(\underset{\sim}{x}_\alpha \in L) = 0$. Then $\text{Prob}(A$ is singular$) = 0$.

Proof:

Note that

$$\det A = N \det.\left[\begin{array}{c|c} \sum_{\alpha=1}^{N} X_\alpha X'_\alpha & \bar{\underset{\sim}{X}} \\ \hline \bar{\underset{\sim}{X}}' & 1/N \end{array}\right] = N \det.\left[\sum_{\alpha=1}^{N} Y_\alpha Y'_\alpha\right]$$

where $\underset{\sim}{Y}'_\alpha = (\underset{\sim}{X}'_\alpha , 1/N)$. Hence

$$A \text{ is singular} \Leftrightarrow \sum_{\alpha=1}^{N} Y_\alpha Y'_\alpha \text{ is singular.}$$

Suppose A is singular with positive probability. Then $(\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_N)$ is of rank $< (p+1)$ with positive probability. Hence (without loss of generality)

$$\det[\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_{p+1}] = 0$$

with positive probability. Note that $p + 1 \leq N$. Thus

$$\begin{aligned} &0 < \text{Prob}\{\det(\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_{p+1}) = 0\} \\ (1) \quad &= E\{\text{Prob}[\det(\underset{\sim}{y}_1, \ldots, \underset{\sim}{y}_p \underset{\sim}{Y}_{p+1}) = 0 | \underset{\sim}{Y}_\alpha = \underset{\sim}{y}_\alpha , \alpha = 1, \ldots, p]\} \end{aligned}$$

where the last coordinate of y_α is $1/N$ $(\alpha = 1, \ldots, p)$. But $\det(\underset{\sim}{y}_1, \ldots, \underset{\sim}{y}_p \underset{\sim}{Y}_{p+1}) = 0$ can be expressed as $\underset{\sim}{b}'\underset{\sim}{X}_{p+1} = a$ where $\underset{\sim}{b}$ and a are functions of $\underset{\sim}{y}_1, \ldots, \underset{\sim}{y}_p$. The conditional probability in (1) is

$$(2) \qquad \text{Prob}[\underset{\sim}{b}'\underset{\sim}{X}_{p+1} = a] ,$$

since $\underset{\sim}{X}_{p+1}$ is independent of $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_p$. (2) is zero unless $\underset{\sim}{b} = \underset{\sim}{0}$ and $a = 0$. Hence, in order that (1) holds, we must have

Theorem 2.

Suppose $X_1, \ldots, X_M$ are mutually independent and for every α $(\alpha = 1, \ldots, M)$ and for every $(p-1)$-dimensional hyperplane L in R^p, $\mathrm{Prob}(X_\alpha \in L) = 0$. Then $\mathrm{Prob}(A$ is singular$) = 0$.

Proof:

Note that

$$\det A = M \det.\left[\begin{array}{c|c} \sum_{\alpha=1}^{M} X_\alpha X_\alpha' & \bar{X} \\ \hline \bar{X}' & 1/M \end{array}\right] = M \det.\left[\sum_{\alpha=1}^{M} Y_\alpha Y_\alpha'\right]$$

where $Y_\alpha' = (X_\alpha' , 1/M)$. Hence

$$A \text{ is singular} \Leftrightarrow \sum_{\alpha=1}^{M} Y_\alpha Y_\alpha' \text{ is singular.}$$

Suppose A is singular with positive probability. Then $(Y_1, \ldots, Y_M)$ is of rank $< (p+1)$ with positive probability. Hence (without loss of generality)

$$\det[Y_1, \ldots, Y_{p+1}] = 0$$

with positive probability. Note that $p + 1 \leq M$. Thus

$$0 < \mathrm{Prob}\{\det(Y_1, \ldots, Y_{p+1}) = 0\}$$

(1) $$= E\{\mathrm{Prob}[\det(y_1, \ldots, y_p Y_{p+1}) = 0 \mid Y_\alpha = y_\alpha , \alpha = 1, \ldots, p]\}$$

where the last coordinate of y_α is $1/M$ $(\alpha = 1, \ldots, p)$. But $\det(y_1, \ldots, y_p Y_{p+1}) = 0$ can be expressed as $b' X_{p+1} = a$ where b and a are functions of $y_1, \ldots, y_p$. The conditional probability in (1) is

(2) $$\mathrm{Prob}[b' X_{p+1} = a] .$$

Since X_{p+1} is independent of $Y_1, \ldots, Y_p$, (2) is zero unless $b = 0$ and $a = 0$. Hence, in order that (1) holds, we must have

(3) $\quad \text{Prob}[\det(\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_p) = 0] > 0$.

Now it follows from the lemma that (3) is false.

Theorem 3.

Let $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_N$ be i.i.d. $p \times 1$ random vectors $(p \leq N-1)$. Then Prob(A is singular) = 0 iff for every $(p-1)$-dimensional hyperplane L in R^p, $\text{Prob}(\underset{\sim}{X}_\alpha \in L) = 0$.

Proof:

Use Theorems 1 and 2.

Further Results.

Let us now consider the question of nonsingularity (with probability 1) of $\underline{X}\, M\, \underline{X}'$ where $\underline{X}$ is defined as before and M is an $N \times N$ p.s.d. matrix.

Let $\text{rank}(M) = r$. Then M can be expressed as

$$M = L_1\ \Lambda\ L_1'$$

where $L_1 : N \times r$, $L_1'L_1 = I_r$ and Λ is a diagonal matrix with all diagonal elements positive. Note now

$$\begin{aligned}\text{rank}(\underline{X}\ M\ \underline{X}') &= \text{rank}(\underline{X}\ L_1\ \Lambda^{1/2}) = \text{rank}(\underline{X}\ L_1) \\ &= \text{rank}(\underline{X}\ L_1L_1'\ \underline{X}') \\ &= \text{rank}[\underline{X}(I_N - L_1L_2')\underline{X}']\end{aligned}$$

where L_2: $N \times (N-r)$, $L_2'L_2 = I_{N-r}$, $L_2'L_1 = 0$. Thus

$$\det[\underline{X}(I_N - L_2L_2')\underline{X}'] = \det[\begin{pmatrix}\underline{X}\\ L_2'\end{pmatrix}(\underline{X}' \quad L_2)] .$$

Hence $\underline{X}\ M\ \underline{X}'$ is singular $\Leftrightarrow$ $\text{rank}\begin{pmatrix}\underline{X}\\ L_2'\end{pmatrix} < p + (N-r)$. It is clear that if $p > r$, $\underline{X}\ M\ \underline{X}'$ is singular. We shall assume now $p \leq r$.

(8) $$\text{Prob}[\det(X_1, \ldots, X_p) = 0] > 0 .$$

Now it follows from the lemma that (8) is false.

Theorem 3.

Let $X_1, \ldots, X_N$ be i.i.d. $p \times 1$ random vectors $(p \leq N - 1)$. Then $\text{Prob}(A \text{ is singular}) = 0$ iff for every $(p-1)$-dimensional hyperplane L in R^p, $\text{Prob}(X_1 \in L) = 0$.

Proof:

Use Theorems 1 and 2.

Further Results.

Let us now consider the question of nonsingularity (with probability 1) of $\bar{X} M \bar{X}'$ where $\bar{X}$ is defined as before and M is an $N \times N$ p.s.d. matrix.

Let $\text{rank}(M) = r$. Then M can be expressed as

$$M = L_1 \Lambda L_1'$$

where $L_1 : N \times r$, $L_1' L_1 = I_r$ and Λ is a diagonal matrix with all diagonal elements positive. Note now

$$\text{rank}(\bar{X} M \bar{X}') = \text{rank}(\bar{X} L_1 \Lambda^{1/2}) = \text{rank}(\bar{X} L_1)$$

$$= \text{rank}(\bar{X} L_1 L_1' \bar{X}')$$

$$= \text{rank}[\bar{X}(I_N - L_2 L_2')\bar{X}']$$

where $L_2 : N \times (N-r)$, $L_2' L_2 = I_{N-r}$, $L_2' L_1 = 0$. Thus

$$\det[\bar{X}(I_N - L_2 L_2')\bar{X}'] = \det[\begin{pmatrix} \bar{X} \\ L_2' \end{pmatrix} \begin{pmatrix} \bar{X}' & L_2 \end{pmatrix}] .$$

Hence $\bar{X} M \bar{X}'$ is singular $\Leftrightarrow$ $\text{rank}\begin{pmatrix} \bar{X} \\ L_2' \end{pmatrix} < p + (N-r)$. It is clear that if $p > r$, $\bar{X} M \bar{X}'$ is singular. We shall assume now $p \leq r$.

Theorem 4.

Suppose there exist (p-1)-dimensional hyperplanes $L_1, \ldots, L_N$ where $L_\alpha : \underset{\sim}{b}'x = a_\alpha$ and $(a_1, \ldots, a_N) \perp C(M)$, where $C(M)$ is the space spanned by the column vectors of M, such that $P[\underset{\sim}{X}_\alpha \in L_\alpha, \alpha = 1, \ldots, N] > 0$, where $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_N$ are $p \times 1$ random vectors. Then

$$P(\underline{\overline{X}}\ M\ \underline{\overline{X}}' \text{ is singular}) > 0.$$

Proof:

$$X_\alpha \in L_\alpha,\ \alpha = 1, \ldots, N \Rightarrow$$

$$b'\underline{\overline{X}}\ M\ \underline{\overline{X}}'b = (a_1, \ldots, a_N)M(a_1, \ldots, a_N)'$$

$$= 0$$

$$\Rightarrow \underline{\overline{X}}\ M\ \underline{\overline{X}}' \text{ is singular.}$$

Theorem 5.

Let $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_N$ be mutually independent $p \times 1$ random vectors. Suppose for every $\alpha(\alpha = 1, \ldots, N)$ and for every (p-1)-dimensional hyperplane L in R^p, $\text{Prob}(\underset{\sim}{X}_\alpha \in L) = 0$. Then, $\text{Prob}(\underline{\overline{X}}\ M\ \underline{\overline{X}}' \text{ is singular}) = 0$.

Proof:

It was shown that

$$\underline{\overline{X}}\ M\ \underline{\overline{X}}' \text{ is singular} \Leftrightarrow \text{rank}\begin{pmatrix}\underline{\overline{X}} \\ L_2'\end{pmatrix} < p + N - r.$$

Without loss of generality, assume

$$L_2' = (Q_1 \quad Q_2 \quad Q_3)$$

where $Q_1 : (N-r) \times (N-r)$ is of rank $N - r$, $Q_2 : (N-r) \times p$. Similarly, partition as

$$\begin{pmatrix}\underline{\overline{X}} \\ L_2'\end{pmatrix} = \begin{pmatrix}\underline{\overline{X}}^{(1)} & \underline{\overline{X}}^{(2)} & \underline{\overline{X}}^{(3)} \\ Q_1 & Q_2 & Q_3\end{pmatrix}$$

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Suppose there exist $(p-1)$-dimensional hyperplanes $L_1, \ldots, L_N$ where $L_\alpha : \underline{b}_\alpha' x = a_\alpha$ and $(a_1, \ldots, a_N) \perp C(M)$, where $C(M)$ is the space spanned by the column vectors of M, such that $P[\underline{X}_\alpha \in L_\alpha, \alpha = 1, \ldots, N] > 0$, where $\underline{X}_1, \ldots, \underline{X}_N$ are $p \times 1$ random vectors. Then

$$P(\underline{X}\, M\, \underline{X}' \text{ is singular}) > 0.$$

Proof:

$$b'\underline{X}\, M\, \underline{X}'b \; L_\alpha,\ \alpha = 1, \ldots, N \Rightarrow$$

$$b'\underline{X}\, M\, \underline{X}'b = (a_1, \ldots, a_N) M (a_1, \ldots, a_N)'$$

$$= 0$$

$$\Rightarrow \underline{X}\, M\, \underline{X}' \text{ is singular.}$$

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Let $\underline{X}_1, \ldots, \underline{X}_N$ be mutually independent $p \times 1$ random vectors. Suppose for every $\alpha (\alpha = 1, \ldots, N)$ and for every $(p-1)$-dimensional hyperplane L in R^p, $\text{Prob}(\underline{X}_\alpha \in L) = 0$. Then, $\text{Prob}(\underline{X}\, M\, \underline{X}' \text{ is singular}) = 0$.

Proof:

It was shown that

$$\underline{X}\, M\, \underline{X}' \text{ is singular} \Leftrightarrow \text{rank} \begin{pmatrix} L_2' \\ \underline{X} \end{pmatrix} < p + N - r.$$

Without loss of generality, assume

$$L_2' = (G_1 \quad G_2 \quad G_3)$$

where $G_1 : (N-r) \times (N-r)$ is of rank $N - r$, $G_2 : (N-r) \times p$. Similarly, partition as

$$\begin{pmatrix} L_2' \\ \underline{X} \end{pmatrix} = \begin{pmatrix} G_1 & G_2 & G_3 \\ \underline{X}_{(1)} & \underline{X}_{(2)} & \underline{X}_{(3)} \end{pmatrix}$$

$$\operatorname{rank}\begin{pmatrix}\bar{\underline{X}} \\ L_2'\end{pmatrix} < p + N - r \Rightarrow \det\begin{bmatrix}\bar{\underline{X}}^{(1)} & \bar{\underline{X}}^{(2)} \\ Q_1 & Q_2\end{bmatrix} = 0$$

$$\Leftrightarrow \det[\bar{\underline{X}}^{(2)} - \bar{\underline{X}}^{(1)}Q_1^{-1}Q_2] = 0.$$

Define

$$Z = (\underset{\sim}{Z}_1, \ldots, \underset{\sim}{Z}_p) = \bar{\underline{X}}^{(2)} - \bar{\underline{X}}^{(1)}Q_1^{-1}Q_2 .$$

Given $\bar{\underline{X}}^{(1)} = x^{(1)}$, $Z_1, \ldots, Z_p$ are mutually independent (conditionally). Moreover, given $\bar{\underline{X}}^{(1)} = x^{(1)}$, probability that any $\underset{\sim}{Z}_\alpha (\alpha = 1, \ldots, p)$ lies in any (p-1)-dimensional hyperplane in R^p is 0. Using the lemma, it is easy to show now

$$\operatorname{Prob}\{\det[\bar{\underline{X}}^{(2)} - \bar{\underline{X}}^{(1)}Q_1^{-1}Q_2] = 0 | \bar{\underline{X}}^{(1)} = x^{(1)}\} = 0.$$

Hence

$$\operatorname{Prob}(\bar{\underline{X}}\, M\, \bar{\underline{X}}' \text{ is singular}) = 0.$$

This result directly follows from the lemma; an alternative proof is given to show a different approach.

Comments.

(1) If $\underset{\sim}{e}_N \perp C(M)$ and $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_N$ are i.i.d., then $\operatorname{Prob}(\bar{\underline{X}}\, M\, \bar{\underline{X}}'$ is singular) = 0 iff for every (p-1)-dimensional hyperplane L in R^p, $\operatorname{Prob}(\underset{\sim}{X}_\alpha \in L) = 0$.

(2) If $p = 1$, and $\bar{\underline{X}}' = (X_1, \ldots, X_N)$ are mutually independent random variables, then $\operatorname{Prob}(\bar{\underline{X}}' M \bar{\underline{X}}$ is singular) = 0 iff $\operatorname{Prob}(\bar{\underline{X}} \perp C(M)) = 0$.

The above results can be shown easily.

$$\operatorname{rank}\begin{pmatrix} L_2' \\ \bar{X} \end{pmatrix} < p + N - r \Leftrightarrow \det\begin{bmatrix} \ell_1' & \ell_2' \\ \bar{X}_{(1)} & \bar{X}_{(2)} \end{bmatrix} = 0$$

$$\Leftrightarrow \det[\bar{X}_{(2)} - \bar{X}_{(1)} L_1^{-1} L_2] = 0 .$$

Define

$$Z = (Z_1, \ldots, Z_p) = \bar{X}_{(2)} - \bar{X}_{(1)} L_1^{-1} L_2 .$$

Given $\bar{X}_{(1)} = x_{(1)}$, $Z_1, \ldots, Z_p$ are mutually independent (conditionally). Moreover, given $\bar{X}_{(1)} = x_{(1)}$, probability that any Z_α $(\alpha = 1, \ldots, p)$ lies in any (p-1)-dimensional hyperplane in R^p is 0. Using the lemma, it is easy to show now

$$\operatorname{Prob}\{\det[\bar{X}_{(2)} - \bar{X}_{(1)} L_1^{-1} L_2] = 0 \mid \bar{X}_{(1)} = x_{(1)}\} = 0 .$$

Hence

$$\operatorname{Prob}(\bar{X} M \bar{X}' \text{ is singular}) = 0 .$$

This result directly follows from the lemma; an alternative proof is given to show a different approach.

Comments.

(1) If $e_N \notin C(M)$ and $X_1, \ldots, X_N$ are i.i.d., then $\operatorname{Prob}(\bar{X} M \bar{X}'$ is singular$) = 0$ iff for every (p-1)-dimensional hyperplane L in R^p, $\operatorname{Prob}(X_\alpha \in L) = 0$.

(ii) If $p = 1$, and $\bar{X}' = (X_1, \ldots, X_N)$ are mutually independent random variables, then $\operatorname{Prob}(\bar{X} M \bar{X}'$ is singular$) = 0$ iff $\operatorname{Prob}(\bar{X} \in C(M)) = 0$.

The above results can be shown easily.

(3) It seems that the negation of the condition in Theorem 5 would be sufficient for $\operatorname{Prob}(\bar{X} M \bar{X}'$ is singular$)$ to be 0 but the present author fails to prove it.

This note was originally written primarily for class-lectures. Dr. M. D. Perlman brought the reference in [1] to my attention and mentioned the problem of studying $\underline{\bar{X}}\, M\, \underline{\bar{X}}'$.

Reference.

[1] Dykstra, R. L. (1970). Establishing the positive-definiteness of the sample covariance matrix. *Ann. Math. Statist.* **41** 2153-2154.

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